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Asymptotically Stable Oscillations in Systems with Hysteresis Nonlinearities

M. Brokate

Mathematisches Seminar, Universität Kiel, 24098 Kiel, Germany
E-mail: mbr@numerik.uni-kiel.de

and

A. V. Pokrovskiĭ*

Institute for Nonlinear Science, Department of Physics, National University of Ireland,

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We present some sufficient conditions for the asymptotic stability of oscillations in nonlinear ODE systems with respect to small hysteretic perturbations. © 1998 Academic Press

Contents.

1. INTRODUCTION

Consider the system of ordinary differential equations

$$x' = f(t, x), \tag{1.1}$$

where $x \in \mathbb{R}^d$ and the right hand side f is T-periodic in t. Suppose that this equation has an asymptotically stable T-periodic solution $x_T = x_T(t)$. It is

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well known that under natural assumptions small perturbations of (1.1) of the form

$$x' = f(t, x) + \varepsilon g(x) \tag{1.2}$$

also have stable T-periodic solutions $x^\varepsilon = x^\varepsilon(t)$ which converge to x_T as $\varepsilon \to 0$, see [5] and the bibliography therein. However, the functional form of the perturbation in (1.2) does not take care of hysteretic perturbations which are rather important in many applications. To cover that case, a more general perturbation, for example of the form

$$x' = f(t, x) + \varepsilon g(x, z(t)), \tag{1.3}$$

$$z(t) = (\Gamma[z_0]\, Lx)(t), \tag{1.4}$$

should be considered. Here, $L\colon \mathbb{R}^d \to \mathbb{R}^m$ is a linear mapping, and $\Gamma[z_0]$ is an operator which transforms functions $u\colon \mathbb{R}_+ \to \mathbb{R}^m$, $\mathbb{R}_+ = [0, +\infty)$, to functions $z\colon \mathbb{R}_+ \to Z$, where Z is a complete metric space equipped with a metric ρ_Z; the argument $z_0 \in Z$ represents the initial memory. As usual, the notation $(\Gamma[z_0]\, u)(t)$ refers to the value of the function $z = \Gamma[z_0]\, u$ at the time t. Although our results are more general, we particularly have in mind such operators Γ which model rate independent hysteretic processes; these operators usually possess specific properties which are discussed in detail, e.g., in [3, 8, 9, 17]. Of those properties, we want to mention at once the *Volterra property*

$$\begin{gathered} u(s) = v(s), \qquad 0 \leqslant s \leqslant t, \qquad \text{implies} \\ (\Gamma[z_0]\, u)(t) = (\Gamma[z_0]\, v)(t), \qquad \text{for all} \quad t \geqslant 0, \end{gathered} \tag{1.5}$$

and the *semi-group property*

$$(\Gamma[(\Gamma[z_0]\, u)(t_1)]\, v)(t_2 - t_1) \equiv (\Gamma[z_0]\, u)(t_2), \tag{1.6}$$

where $v(t) = u(t - t_1)$. Those properties mean that the operators $\Gamma[z_0]$ describe an input-output system where the internal state is equal to the output; of course, more general situations like the Preisach model could also be considered. As it stands here, the system (1.3), (1.4) fits into the framework of the general theory of functional differential Eqs. [6].

Here, we consider specific questions concerning existence and asymptotic stability of periodic solutions of the system (1.3), (1.4) for small ε assuming that the operator Γ has typical "hysteretic" properties. We find it technically convenient to postulate these properties at the beginning, namely in Subsection 2.1. Later in Section 4 we will show that some important hysteresis models indeed have these properties. The general result concerning the asymptotic stability of the system (1.3), (1.4) will be formulated in

Subsection 2.2 and proved in Subsection 2.3. The principal assumptions which we need are a kind of asymptotic stability of the system Γ with respect to the initial condition z_0, namely property (N2) from Subsection 2.1, and a special stability condition for the non-perturbed system (1.1) which combines essential features of exponential stability in the sense of Lyapunov and BIBO (bounded input bounded output) stability in control theory, see Subsection 2.2. In Section 3 we present some applications of our main result to various specific situations.

2. THE MAIN THEOREM

2.1. *Normal Nonlinearities*

Let (Z, ρ_z) be a complete metric space, let $W_t^{1,1}$ be the Banach space of absolutely continuous functions $u: [0, t] \to \mathbb{R}^m$, equipped with the standard norm

$$\|u\|_{W_t^{1,1}} = |u(0)| + \int_0^t |u'(s)|\, ds,$$

define

$$W_{loc}^{1,1} = \{u \mid u: \mathbb{R}_+ \to \mathbb{R}^m,\ u_{|[0,t]} \in W_t^{1,1}\}. \tag{2.1}$$

The nonlinearity Γ appearing in Eq. (1.4), namely $z = \Gamma[z_0]\, u$, is given by an operator

$$\Gamma: W_{loc}^{1,1} \times Z \to C(\mathbb{R}_+; \mathbb{R}^m). \tag{2.2}$$

We assume Γ to possess the *Volterra property* (1.5); therefore, the restrictions $\Gamma_t: W_t^{1,1} \times Z \to C([0, t]; \mathbb{R}^m)$ are well defined, we will also denote them by Γ. We assume that Γ satisfies the following Lipschitz condition:

(N1) There exists a constant $\gamma_u > 0$ such that for every $z_0 \in Z$, every $t \geqslant s \geqslant 0$ and every $u, v \in W_t^{1,1}$ the inequality

$$\rho_z((\Gamma[z_0]\, u)(s), (\Gamma[z_0]\, v)(s)) \leqslant \gamma_u\, \|u - v\|_{W_t^{1,1}} \tag{2.3}$$

holds.

Note that condition (N1) is satisfied for a wide class of hysteresis nonlinearities, see [3, 8, 9, 17]. In contrast to that, the following condition

(N2) is rather specific. It involves a threshold value $h > 0$ and postulates that for any input function u whose *oscillation*

$$\operatorname{osc}_t u = \sup_{0 \leqslant \tau, \sigma \leqslant t} |u(\tau) - u(\sigma)| = \sup_{0 \leqslant \tau \leqslant t} u(\tau) - \inf_{0 \leqslant \tau \leqslant t} u(\tau)$$

on the interval $[0, t]$ exceeds the value h, the corresponding flow $z_0 \mapsto (\Gamma[z_0]\, u)(t)$ becomes a contraction.

(N2) There exists a continuous and bounded function $q: \mathbb{R}_+ \to \mathbb{R}_+$ with $q(\alpha) < 1$ for $\alpha > h$ such that

$$\rho_z((\Gamma[z_0]\, u)(t), (\Gamma[z_1]\, u)(t)) \leqslant q(\operatorname{osc}_t u)\, \rho_z(z_0, z_1) \tag{2.4}$$

holds for all $t \geqslant 0$, all $z_0, z_1 \in Z$ and all $u \in W_t^{1,1}$.

Coupled with the semigroup property (1.6), condition (N2) implies that for T-periodic functions u with $\operatorname{osc}_T u > h$, the distance between z-trajectories belonging to different initial states decreases exponentially with time. For the purpose of this paper, we summarize the requirements concerning Γ in the following definition.

Definition 2.1. (Normal Nonlinearity). A nonlinearity Γ: $W_{loc}^{1,1} \times Z \to C(\mathbb{R}_+; \mathbb{R}^m)$ is called *normal with threshold* $h > 0$ if it satisfies the Volterra property (1.5), the semigroup property (1.6), the Lipschitz condition (N1) and the contraction property (N2) with this value of h.

In Section 4 we will show that some important hysteresis nonlinearities are indeed normal.

2.2. *Formulation of the Main Theorem*

Let the right hand side of the system

$$x' = f(t, x) \tag{2.5}$$

be continuous and Lipschitz continuous in x. Then (2.5) has a unique solution $x = x(t; x_0)$ for any given initial condition $x(0) = x_0$ which moreover depends continuously on x_0. Let x: $\mathbb{R}_+ \to \mathbb{R}^d$ be any solution of (2.5), let $U \subseteq \mathbb{R}^d$ be some neighbourhood of $x(0)$. We say that the solution x is *U-uniformly stable* (cf. [6], Definition 1.1, p. 103), if

$$\lim_{\tau \to \infty} \sup_{x_0 \in U,\, t > \tau} |x(t; x_0) - x(t)| = 0. \tag{2.6}$$

We introduce a similar concept concerning the perturbed system

$$x' = f(t, x) + \varepsilon g(x, z(t)), \tag{2.7}$$

$$z(t) = (\Gamma[z_0]\, Lx)(t), \tag{2.8}$$

where $L\colon \mathbb{R}^d \to \mathbb{R}^m$ is linear. By a solution of (2.7) and (2.8) we mean a classical solution, that is a pair of functions $x\colon \mathbb{R}_+ \to \mathbb{R}^d$ and $z\colon \mathbb{R}_+ \to Z$ where x is continuously differentiable, z is continuous, and (2.7), (2.8) hold at every point $t \geqslant 0$. Recall that (2.7), (2.8) is said to be *well posed* if the corresponding initial value problem with initial conditions

$$x(0) = x_0, \qquad z(0) = z_0, \tag{2.9}$$

has a unique solution

$$(x(t), z(t)) = (x^\varepsilon(t; x_0, z_0), z^\varepsilon(t; x_0, z_0)), \qquad t \geqslant 0 \tag{2.10}$$

for every $x_0 \in \mathbb{R}^d$ and every $z_0 \in Z$, which moreover depends continuously on (x_0, z_0). Let $(x, z)\colon \mathbb{R}_+ \to \mathbb{R}^d \times Z$ be a solution of the well posed system (2.7), (2.8) and U be some neighbourhood of $x(0)$. We say that (x, z) is *U-uniformly stable* if

$$\lim_{\tau \to \infty} \sup_{x_0 \in U,\, z_0 \in Z,\, t > \tau} |x^\varepsilon(t; x_0, z_0) - x(t)| = 0, \tag{2.11}$$

$$\lim_{\tau \to \infty} \sup_{x_0 \in U,\, z_0 \in Z,\, t > \tau} \rho_z(z^\varepsilon(t; x_0, z_0), z(t)) = 0. \tag{2.12}$$

Note that this stability is global with respect to $z_0 \in Z$. This is a natural requirement, because z_0 is connected to the initial state of the perturbation which usually can neither be controlled nor observed. Moreover, we say that (x, z) is *globally asymptotically stable*, if it is U-uniformly stable for each bounded neighbourhood U of $x(0)$.

Let $x_T\colon \mathbb{R}_+ \to \mathbb{R}^d$ be a T-periodic solution of the unperturbed system (2.5) for some period $T > 0$. For the purpose of this paper, we want to define a specific notion of stability of x_T related to the perturbed system

$$x' = f(t, x) + \xi(t), \qquad x(0) = x_0 \tag{2.13}$$

whose unique solution we denote by $x(t; x_0, \xi(\cdot))$, and where $\xi\colon \mathbb{R}_+ \to \mathbb{R}^d$ represents some general continuous perturbation. Namely, we say that the function f is *T-convergent near $x_T(\cdot)$*, if there exist positive numbers ε_c, δ_c,

γ_c, an auxiliary norm $\|\cdot\|_c$ on $\mathbb{R}^d$, and a constant $q_c \in (0, 1)$ such that the relations

$$|x_0 - x_T(0)|, |y_0 - x_T(0)| < \delta_c \qquad \text{and} \qquad |\xi(t)|, |\eta(t)| \leqslant \varepsilon_c, \qquad 0 \leqslant t \leqslant T, \tag{2.14}$$

imply

$$\|x(T; x_0, \xi(\cdot)) - x(T; y_0, \eta(\cdot))\|_c \leqslant q_c \|x_0 - y_0\|_c + \gamma_c \max_{0 \leqslant t \leqslant T} |\xi(t) - \eta(t)|. \tag{2.15}$$

Moreover, we say that f is *globally T-convergent*, if for all $\varepsilon_c, \delta_c > 0$ we can find γ_c, q_c and $\|\cdot\|_c$ as above such that (2.14) and (2.15) holds.

The notion of T-convergence combines essential features of the exponential stability in the sense of Lyapunov and the BIBO (bounded input, bounded output) stability in control theory, see for instance [13, p. 583]. If f is smooth in a neighbourhood of x_T, T-convergence near x_T is equivalent to exponential stability of x_T. In more general situations this property can be extracted from various other stability properties, see Section 3 below.

We also need to impose some growth condition on the perturbation. If f is globally Lipschitz continuous, by virtue of Gronwall's inequality the estimate

$$|x(t; x_0)| \leqslant C(1 + |x_0|), \qquad \forall\, x_0 \in \mathbb{R}^d, \quad \forall\, t \in [0, T], \tag{2.16}$$

holds for the solution of the unperturbed problem with some constant C. In order to obtain a corresponding estimate for the perturbed problem uniformly with respect to z_0, we want the growth condition

$$\text{(G)} \qquad |g(x(t), (\Gamma[z_0]\, Lx)(t))| \leqslant a_g |x(t)| + b_g, \qquad \forall x \in W_T^{1,1}, \qquad \forall z_0 \in Z, \quad \forall t \in [0, T], \tag{2.17}$$

to be satisfied for some constants $a_g, b_g > 0$. A sufficient condition for (2.17) to hold is

$$|g(x, z)| \leqslant a_g |x| + b_g, \qquad \forall x \in \mathbb{R}^d, \quad z \in \mathbb{R}^m. \tag{2.18}$$

If Γ has precompact values in the sense that the sets

$$G_t = \{(\Gamma[z_0]\, u)(s) \colon z_0 \in Z, u \in W_t^{1,1}, s \in [0, t]\} \tag{2.19}$$

are precompact subsets of Z for all $t \geqslant 0$, then it is sufficient to require that

$$|g(x, z)| \leqslant (a\,|x| + b)\,\tilde{g}(z), \qquad \forall\, x \in \mathbb{R}^d, \quad z \in \mathbb{R}^m, \tag{2.20}$$

holds for some continuous function $\tilde{g}$ and some constants $a, b > 0$.

We now formulate the main theorem.

THEOREM 2.1. *Suppose that x_T is a U-uniformly stable T-periodic solution of the system* (1.1) *where f is continuous, satisfies a global Lipschitz condition in x and is T-convergent near x_T. Let g satisfy a global Lipschitz condition in x and z. Let Γ be a normal nonlinearity with the threshold $h > 0$, assume that*

$$osc_t(Lx_T) > h, \tag{2.21}$$

and let, finally, the growth condition (G) *from* (2.17) *be satisfied. Then there exists $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, the perturbed system* (1.3), (1.4) *has a unique T-periodic solution $(x^\varepsilon, z^\varepsilon)$ satisfying $x^\varepsilon(0) \in U$; this solution is U-uniformly stable and enjoys the property*

$$\lim_{\varepsilon \to 0} \sup_{0 \leqslant t \leqslant T} |x_T(t) - x^\varepsilon(t)| = 0. \tag{2.22}$$

Let us discuss this assertion. A method for the investigation of the stability of oscillations in systems with hysteresis nonlinearities was suggested in [18] and further refined in [14]. However, this approach cannot be used to prove Theorem 2.1 for the following two reasons: (i) it requires the operators $\Gamma[z_0]$ to be $(C \mapsto C)$-Lipschitz continuous, (ii) it aims at proving Lyapunov stability rather than asymptotic stability. Concerning the first reason we note that although some hysteresis nonlinearities (in particular in the scalar case $m = 1$) are $(C \mapsto C)$-Lipschitz continuous, some others like the fundamental models to be considered in Section 4, the Duhem model or the model considered by Bliman and Sorine (see, e.g. [1]) etc. do not enjoy this property. Concerning reason (ii), we note that the inequality (2.21) above is essential: very simple examples (see [18]) show that it is unnatural to expect asymptotic stability for oscillations of small amplitude. There is another approach [14] which works for systems with discontinuous nonlinearities, but it requires monotonicity properties in the sense of the theory of semi-ordered spaces (see details in [14], p. 135), which again do not hold for typical nonscalar models. Note also that the methods from [7] can be applied to prove the existence of a periodic solution and to solve some bifurcation problems, but not to analyze asymptotic stability.

Our proof of Theorem 2.1 uses a modified Lyapunov function approach; it is given in Subsection 2.3 below.

Theorem 2.1 can be generalized and modified in various ways as usual. We will give only one useful assertion in this direction. Let $x\colon \mathbb{R}_+ \to \mathbb{R}^d$ be a solution of Eq. (1.1) which is U-uniformly stable with respect to a certain neighbourhood U of $x(0)$. The union of all open neighbourhoods U such that x is U-uniformly stable is called the *basin of attraction* of x.

COROLLARY 2.1. *Suppose that x_T is a uniformly stable T-periodic solution of the Eq.* (1.1) *with respect to some neighbourhood of $x_T(0)$ where f is continuous, satisfies a local Lipschitz condition in x and is T-convergent in a neighbourhood of $x_T(\cdot)$. Let g satisfy a local Lipschitz condition in x and z. Let Γ be a normal nonlinearity with threshold h, let inequality* (2.21) *be valid and the growth condition* (G) *from* (2.17) *be satisfied. Then for any compact subset U of the basin of attraction of x_T, which contains $x_T(0)$ as its interior point, there exists $\varepsilon_0 > 0$ such that the system* (1.3), (1.4) *has for $\varepsilon < \varepsilon_0$ a unique T-periodic solution $(x^\varepsilon, z^\varepsilon)$ satisfying $x^\varepsilon(0) \in U$; this solution is U-uniformly stable and enjoys the property* (2.22).

Proof. Let U be a compact subset of the basin of attraction of x_T which contains $x_T(0)$ as its interior point. Denote by B a closed ball that contains U in its interior. We can as usual redefine the functions f and g such that the new functions coincide with the original ones inside B, are continuous and satisfy corresponding *global* Lipschitz conditions. On the other hand, by compactness of U the solution x_T is U-uniformly stable. The assertions now follow from Theorem 2.1. ∎

2.3. *Proof of Theorem* 2.1

LEMMA 2.1. *For every $\varepsilon \geqslant 0$, the system* (1.3), (1.4) *together with the initial conditions* (2.9) *has a unique solution*

$$(x^\varepsilon, z^\varepsilon) = (x^\varepsilon(t; x_0, z_0), z^\varepsilon(t; x_0, z_0)), \qquad t \geqslant 0. \tag{2.23}$$

Proof. It suffices to establish that the initial value problem

$$x'(t) = (Gx)(t), \qquad x(0) = x_0,$$

where

$$(Gx)(t) = f(t, x) + \varepsilon g(x, (\Gamma[z_0]\, Lx)(t)),$$

has a unique solution $x\colon \mathbb{R}_+ \to \mathbb{R}^d$ for each $z_0 \in Z$. By the Volterra property (1.5), the Lipschitz continuity of the functions f and g in x and by the property (N1), the operator G satisfies for some $\gamma > 0$ the Lipschitz condition

$$|(Gx)(s) - (Gy)(s)| \leqslant \gamma \, \|x - y\|_{W_t^{1,1}}, \qquad x, y \in W_t^{1,1}, \qquad s \in [0, t].$$

Since the associated integral operator $(Fx)(t) = x_0 + \int_0^t (Gx)(s) \, ds$ satisfies

$$\|Fx - Fy\|_{W_t^{1,1}} \leqslant \gamma t \, \|x - y\|_{W_t^{1,1}},$$

the assertion can be proved locally by the standard contraction argument, applied in the space $W_t^{1,1}$ for $t < 1/\gamma$, and globally by continuation, taking into account the semi-group property (1.6). ∎

We now establish some Lipschitz estimates for the dependence of the solutions on the initial condition. The main point is to keep track of the appearance of ε.

LEMMA 2.2. *There exists a continuous function* $\gamma \colon \mathbb{R}_+ \to \mathbb{R}_+$ *such that the estimates*

$$\left| \frac{d}{dt} (x^\varepsilon(t; x_0, z_0) - x^\varepsilon(t; x_1, z_1)) \right| \leqslant \gamma(t)(|x_0 - x_1| + \varepsilon \rho_z(z_0, z_1)), \tag{2.24}$$

$$\left| \frac{d}{dt} (x^\varepsilon(t; x_0, z_0) - x(t; x_1)) \right| \leqslant \gamma(t)(|x_0 - x_1| + \varepsilon(1 + |x_1|)) \tag{2.25}$$

hold for all $t \geqslant 0$, $x_0, x_1 \in \mathbb{R}^d$ and $z_0, z_1 \in Z$.

Proof. Denote $x(t) = x^\varepsilon(t; x_0, z_0)$ and $y(t) = x^\varepsilon(t; x_1, z_1)$. Since x and y solve the perturbed system,

$$\begin{aligned} x'(t) - y'(t) = {} & f(t, x(t)) - f(t, y(t)) + \varepsilon[g(x(t), (\Gamma[z_0] \, Lx)(t)) \\ & - g(y(t), (\Gamma[z_1] \, Ly)(t))]. \end{aligned} \tag{2.26}$$

Therefore,

$$\begin{aligned} |x'(t) - y'(t)| \leqslant {} & (\lambda_f + \varepsilon \lambda_g) \, |x(t) - y(t)| + \varepsilon \lambda_g Q \rho_z(z_0, z_1) \\ & + \varepsilon \lambda_g \gamma_u \gamma_L \left(|x_0 - x_1| + \int_0^t |x'(s) - y'(s)| \, ds \right), \end{aligned}$$

where $Q = \sup_{\alpha \geqslant 0} q(\alpha)$, $\gamma_L = \|L\|$, and λ_f, λ_g are the Lipschitz constants for the functions f and g. Because

$$|x(t) - y(t)| \leqslant |x_0 - x_1| + \int_0^t |x'(s) - y'(s)| \, ds,$$

we obtain from (2.27) that

$$|x'(t) - y'(t)| \leqslant a + b \int_0^t |x'(s) - y'(s)| \, ds,$$

where

$$a = (\lambda_f + \varepsilon \lambda_g \gamma_u \gamma_L) \, |x_0 - x_1| + \varepsilon \lambda_g Q \rho_z(z_0, z_1),$$
$$b = \lambda_f + \varepsilon \lambda_g (1 + \gamma_u \gamma_L).$$

We now obtain (2.24) from Gronwall's inequality, see e.g. [5]. The proof of (2.25) is omitted; it proceeds in the same manner, using the properties (2.16) and (2.17). ∎

COROLLARY 2.2. There exist continuous functions $\gamma_x, \gamma_z : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant γ_w with

$$|x^\varepsilon(t; x_0, z_0) - x^\varepsilon(t; x_1, z_1)| \leqslant \gamma_x(t)(|x_0 - x_1| + \varepsilon \rho_z(z_0, z_1)), \tag{2.28}$$

$$\rho_z(z^\varepsilon(t; x_0, z_0), z^\varepsilon(t; x_1, z_1)) \leqslant \gamma_z(t)(|x_0 - x_1| + \rho_z(z_0, z_1)), \tag{2.29}$$

$$\|x^\varepsilon(\cdot; x_0, z_0) - x^\varepsilon(\cdot; x_1, z_1)\|_{W_t^{1,1}} \leqslant \gamma_w(|x_0 - x_1| + \varepsilon \rho_z(z_0, z_1)), \tag{2.30}$$

$$\|x^\varepsilon(\cdot; x_0, z_0) - x(\cdot; x_1)\|_{W_t^{1,1}} \leqslant \gamma_w(|x_0 - x_1| + \varepsilon(1 + |x_1|)). \tag{2.31}$$

Note that there is no ε in the rightmost term of inequality (2.29).

Proof. Integrating (2.24) we obtain (2.28) with

$$\gamma_x(t) = 1 + \int_0^t \gamma(s) \, ds$$

and (2.30) with

$$\gamma_w = 1 + \int_0^T \gamma(s) \, ds = \gamma_x(T).$$

In the same manner, (2.31) follows from (2.25). Moreover, (2.24) implies

$$\|x^\varepsilon(\cdot; x_0, z_0) - x^\varepsilon(\cdot; x_1, z_1)\|_{W_t^{1,1}} \leqslant (|x_0 - x_1| + \varepsilon \rho_z(z_0, z_1)) \left(1 + \int_0^t \gamma(s) \, ds\right),$$

which, together with the properties (N1) and (N2), yields (2.29) for some suitable function γ_z. ∎

COROLLARY 2.3. *The system* (1.3), (1.4) *is well posed.*

Proof. Follows from Lemma 2.1 and the inequalities (2.28) and (2.29). ▮

We now show that the shift operator

$$Sh^{\varepsilon}\colon (x_0, z_0) \mapsto (x^{\varepsilon}(T; x_0, z_0), z^{\varepsilon}(T; x_0, z_0)) \tag{2.32}$$

is a contraction near $x_T(0)$ for ε small enough. Let us denote by $B_c(\delta)$ the closed δ-ball centered at $x_T(0)$ with respect to the norm $\|\cdot\|_c$ which appears in the Definition (2.14), (2.15) of T-convergence, set

$$q_T = q(\operatorname{osc}_T(Lx_T)), \tag{2.33}$$

recall also that q_c denotes the contraction factor in (2.15).

LEMMA 2.3. *For every q_* satisfying*

$$\max\{q_c, q_T\} < q_* < 1, \tag{2.34}$$

there exist $\delta > 0$, $\varepsilon_0 > 0$ and a metric ρ_{q*} on $B_c(\delta) \times Z$ such that for every $0 < \varepsilon < \varepsilon_0$ the shift operator Sh^{ε} defined in (2.32) *becomes a q_*-contraction with respect to ρ_q which maps $B_c(\delta) \times Z$ into itself.*

Proof. Since $x_T(0) = x(T; x_T(0))$, we see from the definition of T-convergence with $\xi(t) = \varepsilon g(x^{\varepsilon}(t), z^{\varepsilon}(t))$ and $\eta \equiv 0$ that

$$\|x^{\varepsilon}(T; x_0, z_0) - x_T(0)\|_c \leqslant q_c \|x_0 - x_T(0)\|_c + \gamma_c \varepsilon \sup_{t \in [0, T]} |g(x^{\varepsilon}(t), z^{\varepsilon}(t))| \tag{2.35}$$

holds on $B_c(\delta) \times Z$ if $\delta < \delta_c$ and $\varepsilon \leqslant \varepsilon_1$, where ε_1 is chosen such that

$$\varepsilon_1 \sup_{t \in [0, T],\, \varepsilon \leqslant \varepsilon_1} |g(x^{\varepsilon}(t), z^{\varepsilon}(t))| \leqslant \varepsilon_c,$$

the supremum being finite because of the bounds in Corollary 2.2 and the growth condition (G). Choosing $\varepsilon_2 < \varepsilon_1$ small enough, we conclude that the shift operator Sh^{ε} maps $B_c(\delta) \times Z$ into itself if $\delta < \delta_c$ and $\varepsilon < \varepsilon_2$. To derive the contraction property, let $(x_0, z_0), (x_1, z_1) \in B_c(\delta) \times Z$ be given, let us introduce the abbreviations

$$x(t) = x^{\varepsilon}(t; x_0, z_0), \qquad y(t) = x^{\varepsilon}(t; x_1, z_1) \qquad \text{and}$$
$$z(t) = z^{\varepsilon}(t; x_0, z_0), \qquad w(t) = z^{\varepsilon}(t; x_1, z_1).$$

Then

$$x'(t) = f(t, x(t)) + \xi(t), \qquad y'(t) = f(t, y(t)) + \eta(t)$$

with

$$\xi(t) = \varepsilon g(x(t), z(t)), \qquad \eta(t) = \varepsilon g(y(t), w(t))$$

holds for all $t \in [0, T]$. Corollary 2.2 implies

$$|\xi(t) - \eta(t)| \leqslant \varepsilon \gamma_*(|x_0 - x_1| + \rho_z(z_0, z_1)), \qquad t \in [0, T],$$

with some constant $\gamma_* > 0$. By the definition of T-convergence

$$\|x(T) - y(T)\|_c < q_c \|x_0 - x_1\|_c + \gamma_c \gamma_* \varepsilon(|x_0 - x_1| + \rho_z(z_0, z_1)) \tag{2.36}$$

holds if $\varepsilon \leqslant \varepsilon_3$, where ε_3 is chosen such that

$$\varepsilon_3 \gamma_*(|x_0 - x_1| + \rho_z(z_0, z_1)) \leqslant \varepsilon_c. \tag{2.37}$$

To derive a corresponding estimate for $\rho_z(z(T), w(T))$, we use property (N2) as follows. We first claim that

$$q(\operatorname{osc}_T(Lx^\varepsilon(\cdot; x_0, z_0))) \leqslant q_T + \beta(\varepsilon, \delta),$$

where β is a certain function with $\lim_{\varepsilon, \delta \to 0} \beta(\varepsilon, \delta) = 0$. Indeed, this follows from (2.31) with $x_1 = x_T(0)$ (thus $x(\cdot; x_1) = x_T$), and from the continuity of q. Next, the use of the triangle inequality as well as of properties (N1) and (N2) yields

$$\rho_z(z(T), w(T)) \leqslant \gamma_u \gamma_L \|x - y\|_{W_T^{1,1}} + (q_T + \beta(\varepsilon, \delta))\, \rho_z(z_0, z_1). \tag{2.39}$$

Due to the equivalence of norms in $\mathbb{R}^d$ we conclude from (2.36) and (2.39), using again the estimates of Corollary 2.2, that there exist constants $\gamma_1, \gamma_2, \gamma_3$ not depending on ε with

$$\|x(T) - y(T)\|_c \leqslant (q_c + \varepsilon\gamma_1) \|x_0 - x_1\|_c + \varepsilon \rho_z(z_0, z_1), \tag{2.40}$$

$$\rho_z(z(T), w(T)) \leqslant \gamma_2 \|x_0 - x_1\|_c + (q_T + \beta(\varepsilon, \delta) + \varepsilon\gamma_3)\, \rho_z(z_0, z_1). \tag{2.41}$$

An explicit inspection of the characteristic equation of the matrix

$$A_\varepsilon = \begin{pmatrix} q_c + \varepsilon\gamma_1 & \varepsilon \\ \gamma_2 & q_T + \beta(\varepsilon, \delta) + \varepsilon\gamma_3 \end{pmatrix} \tag{2.42}$$

shows that its spectral radius $r(A_\varepsilon)$ satisfies

$$r(A_\varepsilon) = \max\{q_c, q_T\} + \alpha(\varepsilon, \delta), \qquad \lim_{\varepsilon, \delta \to 0} \alpha(\varepsilon, \delta) = 0. \tag{2.43}$$

Now choose a norm $\|\cdot\|_*$ on $\mathbb{R}^2$ such that $\|A_\varepsilon\|_* \leqslant q_*$ holds for the associated operator norm, and define the metric ρ_{q_*} on $B_c(\delta) \times Z$ by

$$\rho_{q_*}((x_0, z_0), (x_1, z_1)) = \|(\|x_0 - x_1\|_c, \rho_z(z_0, z_1))\|_*. \tag{2.44}$$

Then the estimates (2.41) and (2.42) show that

$$\rho_{q_*}((x(T), z(T)), (y(T), w(T))) \leqslant q_* \rho_{q*}((x_0, z_0), (x_1, z_1)). \tag{2.45}$$

As the choice of (x_0, z_0) and (x_1, z_1) was arbitrary, the lemma is proved. ▮

Now we can complete the proof of the theorem. Lemma 2.3 and the contraction mapping principle show that the mapping Sh^ε has a unique fixed point $(x_0^\varepsilon, z_0^\varepsilon)$ in $B_c(\delta) \times Z$. This fixed point defines a T-periodic solution $(x^\varepsilon, z^\varepsilon)$ of the system (1.3), (1.4) by

$$x^\varepsilon(t) = x(t; x_0^\varepsilon, z_0^\varepsilon), \qquad z^\varepsilon(t) = z(t; x_0^\varepsilon, z_0^\varepsilon), \qquad 0 < \varepsilon < \varepsilon_0.$$

It is easy to check from the estimates of Corollary 2.2 that this periodic solution is unique with respect to initial values in $B_c(\delta)$ and that it is $B_c(\delta)$-uniformly stable. These properties extend to U by the following argument. Since x_T is U-uniformly stable,

$$\lim_{n \to \infty} \sup_{x_0 \in U} |x(nT; x_0) - x_T(0)| = 0. \tag{2.46}$$

We therefore find an N such that $x(NT; x_0) \in B_c(\delta/2)$ for all $x_0 \in U$. Moreover, (2.31) implies that

$$\lim_{\varepsilon \to 0} \sup_{x_0 \in U,\, z_0 \in Z} |x^\varepsilon(t; x_0, z_0) - x(t; x_0)| = 0 \tag{2.47}$$

holds for all $t \geqslant 0$. Thus, for ε small enough we have $x^\varepsilon(NT; x_0, z_0) \in B_c(\delta)$ for all $x_0 \in U$ and all $z_0 \in Z$. Thus, the $B_c(\delta)$-uniform stability of $(x^\varepsilon, z^\varepsilon)$ implies that

$$\lim_{\tau \to \infty} \sup_{x_0 \in U,\, z_0 \in Z,\, t > \tau} |x^\varepsilon(t + NT; x_0, z_0) - x^\varepsilon(t)| = 0 \tag{2.48}$$

as well as

$$\lim_{\tau \to \infty} \sup_{x_0 \in U,\, z_0 \in Z,\, t > \tau} \rho_z(z^\varepsilon(t + NT; x_0, z_0), z^\varepsilon(t)) = 0. \tag{2.49}$$

The proof is complete. ▮

3. APPLICATIONS

3.1. *Global Stability*

In this subsection, we consider the special case

$$f(t, x) = Ax + bF(t, c^T x) \tag{3.1}$$

of (1.1), where $A \in \mathbb{R}^{d,d}$, $b, c \in \mathbb{R}^d$, and the function $F: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is T-periodic in t. (Here and in the following, ^{T} denotes the transpose of a vector respectively a matrix.) This equation arises for example in control theory when we use a nonlinear feedback $u = F(t, y)$ for the SISO (single input single output) control system $x' = Ax + bu$, $y = c^T x$, see e.g. [12].

We want to investigate the global asymptotic stability (which we have defined as U-uniform stability for all bounded neighbourhoods, see Section 2) of the perturbed system (1.3), (1.4) which now reads as

$$x' = Ax + bF(t, c^T x) + \varepsilon g(x, z(t)), \tag{3.2}$$

$$z(t) = (\Gamma[z_0]\, Lx)(t). \tag{3.3}$$

Suppose that A is a stable matrix satisfying the frequency inequality and that

$$\lambda_F \, \|G\| < 1$$

holds, where λ_F is a Lipschitz constant for F in y, and

$$\|G\| = \max_{-\infty < \omega < \infty} |G(i\omega)| = \max_{-\infty < \omega < \infty} |c^T (i\omega - A)^{-1} b| \tag{3.5}$$

denotes the operator norm of the transfer function G of the linear system (A, b, c^T) in the frequency domain. Under these conditions the unperturbed Eq. (1.1) has a unique T-periodic solution x_T which is globally asymptotically stable [12]. The following theorem shows that the same is true for the perturbed system (3.2), (3.3).

THEOREM 3.1. *Suppose that the functions F and g are continuous, that* (2.20) *holds and that g satisfies a global Lipschitz condition in x and z. Let the nonlinearity Γ be normal and the inequality* (2.21) *be valid, let the growth condition* (G) *hold. Then there exists $\varepsilon_0 > 0$ such that the system* (3.2), (3.3) *has a unique T-periodic solution $(x^\varepsilon, z^\varepsilon)$ for every $0 < \varepsilon < \varepsilon_0$. This solution is globally asymptotically stable and enjoys the property* (2.22).

The remainder of this subsection is devoted to the proof of Theorem 3.1. It is well known (see for instance [12], Lemma 6, p. 124) that condition

(2.20) implies the existence of a symmetric positive definite matrix $P \in \mathbb{R}^{d,d}$ and a number $\alpha > 0$ such that the matrix Lyapunov equation

$$A^T P + PA = -qq^T - \alpha I - \lambda_F^2 cc^T, \qquad q = Pb, \tag{3.6}$$

is satisfied. We introduce the quadratic Lyapunov function

$$v(x) = \tfrac{1}{2} x^T P x$$

together with its corresponding norm

$$\|x\|_P = \sqrt{v(x)}.$$

By $x(t; x_0, \xi(\cdot))$ we denote the solution of the initial value problem

$$x' = Ax + bF(t, c^T x) + \xi(t), \qquad x(0) = x_0.$$

LEMMA 3.1. *There exist $\alpha_0 > 0$ and $\gamma > 0$ such that the inequality*

$$\|x(t; x_0, \xi(\cdot)) - x(t; y_0, \eta(\cdot))\|_P \leqslant e^{-\alpha_0 t} \|x_0 - y_0\|_P + \gamma \sup_{0 \leqslant t \leqslant T} |\xi(t) - \eta(t)|. \tag{3.7}$$

holds for all $x_0, y_0 \in \mathbb{R}^d$ and all continuous functions $\xi, \eta : [0, T] \to \mathbb{R}^d$. In particular, the right hand side $f(t, x) = Ax + bF(t, c^T x)$ is globally T-convergent.

This assertion, as well as the assertion of the Lemma 3.2 below, are well known. We nevertheless sketch their proofs for the convenience of the reader.

Proof. The difference

$$r(t) = x(t; x_0, \xi(\cdot)) - x(t; y_0, \eta(\cdot))$$

satisfies

$$r'(t) = Ar(t) + b\varphi(t) + \xi(t) - \eta(t) \tag{3.8}$$

where

$$|\varphi(t)| \leqslant \lambda_F |c^T r(t)|. \tag{3.9}$$

We compute the derivative of the function $t \mapsto v(r(t))$ with the chain rule and use (3.6) and (3.8) to arrive at

$$\frac{d}{dt} v(r(t)) = -\frac{1}{2}\left((r(t)^T q - \varphi(t))^2 + \lambda_F^2 (c^T r(t))^2 - \varphi(t)^2 + \alpha r(t)^T r(t)\right)$$
$$+ r(t)^T P(\xi(t) - \eta(t)).$$

Because of (3.9) it follows that

$$\frac{d}{dt} v(r(t)) \leqslant -\alpha\, |r(t)|^2 + r(t)^T P(\xi(t) - \eta(t)).$$

This can be rewritten as

$$\frac{d}{dt} \|r(t)\|_P \leqslant -c_1 \|r(t)\|_P + c_2 \max |\xi(t) - \eta(t)|.$$

The first assertion of the lemma now follows from Gronwall's inequality, while the second is an immediate consequence of the first. ∎

Denote $B(R) = \{x \in \mathbb{R}^d : |x| \leqslant R\}$. By Lemma 3.1 and Theorem 2.1 applied with $U = B(R)$, we see that for each $R > |x_T(0)|$ there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$, the system (3.2), (3.3) has a unique T-periodic solution $(x^\varepsilon, z^\varepsilon)$ satisfying $|x^\varepsilon(0)| < R$; this solution is $B(R)$-uniformly stable and enjoys the property (2.22). To complete the proof of the theorem it remains to show that ε_0 can be chosen uniformly for all $R > 0$; this will be accomplished if there exist $R_* > 0$, $\varepsilon_* > 0$ and a function $: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$x^\varepsilon(t; x_0, z_0) \leqslant R_*$$

for all

$$\varepsilon < \varepsilon_*, \qquad x_0 \in \mathbb{R}^d, \qquad z_0 \in Z, \qquad t \geqslant \tau(|x_0|).$$

This follows, for instance, from the following analog of Lemma 3.1.

Lemma 3.2. *For each $\delta > 0$, $R > 0$, there exist $\tau > 0$ and $\varepsilon_0 > 0$ such that the inequality*

$$\|x^\varepsilon(t; x_0, z_0) - x_T(t)\|_P < \delta \tag{3.10}$$

holds for all $|x_0| < R$, $z_0 \in Z$, $\varepsilon < \varepsilon_0$ and $t > \tau$.

Proof. Due to the growth condition (G), the function

$$r(t) = x^{\varepsilon}(t; x_0, z_0) - x_T(t)$$

satisfies the equation

$$r'(t) = Ar(t) + b\varphi(t) + \zeta(t),$$

where

$$|\varphi(t)| \leqslant \lambda_F \, |c^T r(t)|, \qquad |\zeta(t)| \leqslant \varepsilon(a_1 \, |r(t)| + b_1). \tag{3.11}$$

With an argument analogous to that in the proof of the preceding lemma we arrive at the inequality

$$\frac{d}{dt} \, \|r(t)\|_P \leqslant -c_1 \, \|r(t)\|_P + \varepsilon(a_2 \, |r(t)| + b_2),$$

and again we only have to use Gronwall's inequality to prove the lemma and thus to complete the proof of Theorem 3.1. ∎

Other kinds of frequency criteria (see, e.g., [12]) can be used in a similar way.

3.2. *Local Stability*

Let us return to the general perturbed system (1.3), (1.4). Suppose that the function f is smooth and that the unperturbed system (1.1) has a T-periodic solution $x_T(t)$. We consider the linearization of the system (1.1) along the periodic trajectory x_T:

$$y' = A(t) \, y, \qquad A(t) = \partial_x f(t, x_T(t)). \tag{3.12}$$

Denote by S the set of characteristic multipliers ([5], p. 237) of the linear periodic system (3.12).

Proposition 3.1. *Suppose that S belongs to the open unit disc of the complex plane and that g is continuous and satisfies a local Lipschitz condition in x and z. Let the nonlinearity Γ be normal, let the inequality $\mathrm{osc}_T L x_T(\cdot) > h$ hold, and let the growth condition* (G) *be satisfied. Then for any compact subset U of the basin of attraction of x_T, which contains $x_T(0)$ as its interior point, there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$, the system* (1.3), (1.4) *has a unique T-periodic solution $(x^{\varepsilon}, z^{\varepsilon})$ satisfying $x^{\varepsilon}(0) \in U$; this solution is U-uniformly stable and enjoys the property* (2.22).

Proof. By Corollary 2.1 it suffices to show that the mapping f is T-convergent in a neighbourhood of x_T. The set of characteristic multipliers coincides with the set of eigenvalues of the shift operator Sh_* along trajectories of the linear system (3.12) [5]. Consequently, there exists a norm $\|\cdot\|_c$ in $\mathbb{R}^d$ such that the shift operator Sh_* is a q_*-contraction in this norm with some $q_* < 1$ (see Proposition 9.6, [4], p. 83).

Choose any $q_c \in (q_*, 1)$. We have to prove that there exist positive ε_c, δ_c, γ_c such that the relations

$$|x_0 - x_T(0)|,\ |y_0 - x_T(0)| < \delta_c \qquad \text{and} \qquad |\xi(t)|,\ |\eta(t)| \leqslant \varepsilon_c, \quad 0 \leqslant t \leqslant T \tag{3.13}$$

imply

$$\|x(T, x_0, \xi(\cdot)) - x(T, y_0, \eta(\cdot))\|_c \leqslant q_c \|x_0 - y_0\|_c + \gamma_c \max_{0 \leqslant t \leqslant T} |\xi(t) - \eta(t)|. \tag{3.14}$$

To this end, we observe that the difference

$$r(t) = x(t, x_0, \xi(\cdot)) - x(t, y_0, \eta(\cdot)) \tag{3.15}$$

of the solutions of the two initial value problems

$$x' = f(t, x) + \xi(t), \qquad x(0) = x_0$$

and

$$y' = f(t, y) + \eta(t), \qquad y(0) = y_0$$

satisfy

$$r'(t) = A(t)\, r(t) + \xi(t) - \eta(t) + o(r(t)),$$

if we choose δ_c, ε_c in (3.13) sufficiently small. As before we conclude that

$$\|r(T)\|_c \leqslant q_c \|r(0)\|_c + \gamma_c \max_{0 \leqslant t \leqslant T} |\xi(t) - \eta(t)| \tag{3.16}$$

holds for sufficiently small δ_c, ε_c and for a suitable constant γ_c, which yields (3.14). The lemma is proved. ▮

Observe that the assumption that S belongs to the open unit disc of the complex plane means just that the solution $x_T(\cdot)$ is supposed to be *exponentially stable* [10], that is, there is a $\gamma > 0$ and, for each $\varepsilon > 0$, a $\delta(\varepsilon) > 0$, such that $|x_0 - x_T(0)| < \delta$ implies $|x(t; x_0) - x_T(t)| < \varepsilon e^{-\gamma t}$, for $t > 0$.

3.3. *Self-Oscillations*

Let us briefly consider the autonomous system

$$x' = f(x), \tag{3.17}$$

let us assume f to be globally Lipschitz continuous. The perturbed system becomes

$$x' = f(x) + \varepsilon g(x, z(t)), \tag{3.18}$$

$$z(t) = (\Gamma[z_0]\, Lx)(t). \tag{3.19}$$

Let $x_*(\cdot)$ be a periodic solution of (3.17) with the minimal period $T>0$. Denote by $X \subset \mathbb{R}^d$ the orbit of this solution, that is,

$$X = \{x_*(t) : 0 \leqslant t \leqslant T\}.$$

Let $U \subseteq \mathbb{R}^d$ be some neighbourhood of $x_*(0)$. We assume that the solution x_* is *U-uniformly orbitally stable*, that is,

$$\lim_{\tau \to \infty} \sup_{x_0 \in U,\, t > \tau} \rho(x(t; x_0), X) = 0$$

holds, where

$$\rho(x, X) = \inf_{y \in X} |x - y|, \qquad x \in \mathbb{R}^d.$$

Analogously, the solution $\bar{x}(t) = (x_*(t), z(t))$ of the well posed system (3.18), (3.19) is said to be *U-uniformly orbitally stable* if

$$\lim_{\tau \to \infty} \sup_{x_0 \in U,\, z_0 \in Z,\, t > \tau} \rho(\bar{x}^\varepsilon(t; x_0, z_0), \bar{X}) = 0, \tag{3.20}$$

where

$$\bar{x}^\varepsilon(t; x_0, z_0) = (x^\varepsilon(t; x_0, z_0), z^\varepsilon(t; x_0, z_0))$$

and $\rho((x, z), \bar{X}) = \max_{(y, w) \in \bar{X}} \{\min\{|x - y|, \rho_z(z, w)\}\}$.

Let Π be a hyperplane in $\mathbb{R}^d$ which contains $x_*(0)$ and is transversal to the vector $f(x_*(0))$. If $\delta > 0$ is small enough, the Poincaré operator

$$P_\Pi : \Pi \cap B(x_*(0), \delta) \to \Pi, \tag{3.21}$$

with

$$P_\Pi(x_0) = x(t; x_0), \qquad t = \min\{s \mid s > 0,\, x(s; x_0) \in \Pi\}, \tag{3.22}$$

is well defined. Let us again denote by $x(t; x_0, \xi(\cdot))$ the unique solution of the initial value problem

$$x' = f(x) + \xi(t), \qquad x(0) = x_0, \tag{3.23}$$

where $\xi(\cdot)$ is a given continuous perturbation. The function f is said to be *Π-convergent near x_** if there exist positive constants ε_c, δ_c, γ_c, an auxiliary norm $\|\cdot\|_c$ on $\mathbb{R}^{d-1}$ and $q_c \in (0, 1)$ such that the relations

$$x_0, y_0 \in \Pi, \qquad \|x_0 - x_*(0)\|_c, \|y_0 - x_*(0)\|_c < \delta_c, \quad |\xi(t)|, |\eta(t)| \leqslant \varepsilon_c, \quad t \geqslant 0 \tag{3.24}$$

imply

$$\|P_\Pi(x_0, \xi(\cdot)) - P_\Pi(y_0, \eta(\cdot))\|_c \leqslant q_c \|x_0 - y_0\|_c + \gamma_c \max_{0 \leqslant t \leqslant T} |\xi(t) - \eta(t)|. \tag{3.25}$$

(In (3.25), P_Π denotes the Poincaré operator corresponding to the perturbed system (3.23).)

THEOREM 3.2. *Suppose that x_* is a U-uniformly orbitally T-periodic solution of the Eq.* (3.17), *where f satisfies a global Lipschitz condition and is Π-convergent near x_*. Let g satisfy a global Lipschitz condition in x and z. Let the nonlinearity Γ be normal and satisfy the inequality* (2.21), *and let the growth condition* (G) *hold. Then there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$, the system* (3.18), (3.19) *has a periodic solution $\bar{x}^\varepsilon = (x^\varepsilon, z^\varepsilon)$ with the minimal period T^ε satisfying $x^\varepsilon(0) \in U$; this solution is U-uniformly orbitally stable and unique except for translation in time. Its* (*unique*) *orbit $\bar{X}^\varepsilon$ enjoys the properties*

$$\lim_{\varepsilon \to 0} \rho_H(\bar{X}_*, \bar{X}^\varepsilon) = 0, \qquad \lim_{\varepsilon \to 0} T^\varepsilon = T, \tag{3.26}$$

where ρ_H is the Hausdorff metric

$$\rho_H(\bar{X}, \bar{Y}) = \max\{\max_{\bar{x} \in \bar{X}} \rho(\bar{x}, \bar{Y}), \max_{\bar{y} \in \bar{Y}} \rho(\bar{y}, \bar{X})\}.$$

The proof is similar to that of Theorem 2.1 and is therefore omitted.

Let again x_* be a periodic solution of (3.17). The union of all open neighbourhoods U such that x_* is U-uniformly orbitally stable is called the *basin of orbital attraction* of x_*.

COROLLARY 3.1. *Suppose that x_* is a uniformly orbitally stable T-periodic solution of the Eq.* (1.1) *with respect to some neighbourhood of x_*. Let f*

satisfy a local Lipschitz condition and be Π-convergent near x_*. Let g satisfy a local Lipschitz condition in x and z. Let the nonlinearity Γ be normal and satisfy the inequality (2.21), *let the growth condition* (G) *be satisfied. Suppose that U is a compact subset of basin of attraction of x_* which contains $x_*(0)$ as its interior point. Then there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$, the system* (3.18), (3.19) *has a T^ε-periodic solution* $(x^\varepsilon, z^\varepsilon)$ *satisfying $x^\varepsilon(0) \in U$; this solution is U-uniformly orbitally stable and unique except for translation in time; its orbit enjoys the properties* (3.26).

Similarly to Proposition 3.1, the following proposition can also be proved. Suppose that the function f is smooth and that the unperturbed Eq. (1.1) has a T-periodic solution $x_*(t)$.

PROPOSITION 3.2. *Suppose that x_T is a uniformly orbitally stable T-periodic solution of the Eq.* (1.1) *with respect to some neighbourhood of $x_T(0)$. Suppose that the set of characteristic multipliers* ([16], *p.* 167) *of its orbit belongs to the open unit disc of the complex plane, and that g is continuous and satisfies a local Lipschitz condition in x and z. Let the nonlinearity Γ be normal and the inequality* (2.21) *be satisfied. Suppose that U is a compact subset of the basin of attraction of x_T which contains $x_T(0)$ as its interior point. Then there exists $\varepsilon_0 > 0$ such that the system* (3.18), (3.19) *has for $0 < \varepsilon < \varepsilon_0$ a T^ε-periodic solution $\bar{x}(\cdot) = (x^\varepsilon(\cdot), z^\varepsilon(\cdot))$ satisfying $x_T(0) \in U$; this solution is U-uniformly orbitally stable and unique except for translation in time, and its orbit enjoys the properties* (3.26).

4. EXAMPLES OF NORMAL NONLINEARITIES

Let $Z \subset \mathbb{R}^m$ be a closed convex bounded set in $\mathbb{R}^m$ with nonempty interior int Z, let $\langle \cdot, \cdot \rangle$ denote the standard scalar product in $\mathbb{R}^m$. For a given $z_0 \in Z$ we define the operator $\mathcal{S}_Z[z_0]: W_T^{1,1} \to W_T^{1,1}$ as the operator which maps any function $u \in W_T^{1,1}$ to the solution z of the variational inequality

$$\langle \dot{z}(t) - \dot{u}(t)\, z(t) - \tilde{z} \geqslant 0 \quad \forall \tilde{z} \in Z, \qquad \text{a.e. in } [0, T], \tag{4.1}$$

$$z(t) \in Z, \qquad \forall t \in [0, T], \tag{4.2}$$

satisfying the initial condition

$$z(0) = z_0. \tag{4.3}$$

The operators $\mathscr{S}_Z$ describe the so-called *stop* nonlinearity, and the operators $\mathscr{P}_Z$ defined by

$$\mathscr{P}_Z[z_0]\,u = u - \mathscr{S}_Z[z_0]\,u \tag{4.4}$$

describe the *play* nonlinearity. The set Z is called the *characteristic* of $\mathscr{S}_Z$ and $\mathscr{P}_Z$. The definition of the operators $\mathscr{S}_Z[z_0]$ and $\mathscr{P}_Z[z_0]$ is meaningful; since the fundamental work of Brézis and Moreau in the sixties, the system (4.1)–(4.3) is known to have a unique solution. These operators have been studied to some extent in the monographs [8, 9]. Among other applications, they serve as basic elements from which many constitutive laws in elastoplasticity can be constructed, both in the convex (see, e.g. [11]) and in the non-convex case [2].

The nonlinearity $\mathscr{S}_Z$, and therefore also $\mathscr{P}_Z$ satisfies condition (N1). Indeed, the basic uniqueness and stability argument (test the variational inequalities for $z = \mathscr{S}_Z[z_0]\,u$ and for $w = \mathscr{S}_Z[z_1]\,v$ with $(z+w)/2$) yields

$$|(\mathscr{S}_Z[z_0]\,u)(t) - (\mathscr{S}_Z[z_1]\,v)(t)| \leqslant |z_0 - z_1| + \int_0^t |\dot{u}(s) - \dot{v}(s)|\,ds. \tag{4.5}$$

The property (N2) is not always satisfied for $\mathscr{S}_Z$; for example, if Z is chosen as the cube $[-1, 1]^m$, z_0 and z_1 lie on a common face, and $\dot{u}$ always points in the normal direction to that face, then the distance (in fact, the difference) between $\mathscr{S}_Z[z_0]\,u$ and $\mathscr{S}_Z[z_1]\,u$ will remain constant. On the other hand, the following variant of strict convexity will turn out to be sufficient for (N2) to hold.

Let $\gamma > 0$. The set Z is called *γ-convex*, if for every $z_0, z_1 \in Z$ the ball with radius $\gamma\,|z_0 - z_1|^2$ centered at $(z_0 + z_1)/2$ is contained in Z.

PROPOSITION 4.1. *Let Z be γ-convex. Then the stop nonlinearity $\mathscr{S}_Z$ is normal with the threshold $h = \operatorname{diam}(Z)$. In particular, property* (N2) *holds with q given by*

$$q_Z(\alpha) = \min\{e^{-4\gamma(\alpha - \operatorname{diam}(Z))}, 1\}. \tag{4.6}$$

Proof. Because of (4.5), property (N1) holds and it suffices to show that

$$|(\mathscr{S}_Z[z_0]\,u)(t) - (\mathscr{S}_Z[z_1]\,u)(t)| \leqslant |z_0 - z_1|\,e^{-4\gamma(\operatorname{osc}_t(u) - \operatorname{diam}(Z))} \tag{4.7}$$

holds for all $u \in W_T^{1,1}$, all $z_0, z_1 \in Z$ and all $t \in [0, T]$. To this end, we first define $d\colon Z \times Z \to \mathbb{R}_+$ by

$$d(z_0, z_1) = \inf_{\zeta \in \partial Z} \left| \frac{z_0 + z_1}{2} - \zeta \right|, \tag{4.8}$$

where ∂Z denotes the boundary of Z. Let us now fix $u \in W_T^{1,1}$ and $z_0, z_1 \in Z$, let us denote

$$z(t) = (\mathscr{S}_Z[z_0]\, u)(t), \qquad w(t) = (\mathscr{S}_Z[z_1]\, u)(t) \tag{4.9}$$

and

$$\xi(t) = (\mathscr{P}_Z[z_0]\, u)(t), \qquad \eta(t) = (\mathscr{P}_Z[z_1]\, u)(t). \tag{4.10}$$

From the variational inequalities which define z and w it follows that

$$\frac{d}{dt}\frac{1}{4}\,|z(t) - w(t)|^2 \leqslant -d(z(t), w(t)) \cdot (|\dot{\xi}(t)| + |\dot{\eta}(t)|). \tag{4.11}$$

(In order not to interrupt the main flow of the argument, the proof of (4.11) is given below in Lemma 4.1.) Now, since Z is γ-convex, (4.8) implies

$$d(z(t), w(t)) \geqslant \gamma\, |z(t) - w(t)|^2. \tag{4.12}$$

Inequalities (4.11) and (4.12) together imply that

$$\frac{d}{dt}\,|z(t) - w(t)| \leqslant -2\gamma(|\dot{\xi}(t)| + |\dot{\eta}(t)|) \cdot |z(t) - w(t)|. \tag{4.13}$$

On the other hand, we always have

$$\int_0^t |\dot{\xi}(s)|\, ds = = : \operatorname{var}_t \xi \geqslant \operatorname{osc}_t \xi \geqslant \operatorname{osc}_t(u) - \operatorname{diam}(Z)$$

and

$$\operatorname{var}_t \eta \geqslant \operatorname{osc}_t \eta \geqslant \operatorname{osc}_t(u) - \operatorname{diam}(Z).$$

Therefore, (4.13) implies

$$\frac{d}{dt}\,|z_1(t) - z_2(t)| \leqslant -4\gamma(\operatorname{osc}_t(u) - \operatorname{diam}(Z)) \cdot |z(t) - w(t)|$$

and (4.7) follows. The proposition is proved. ∎

LEMMA 4.1. *The functions z, w, ξ, η defined in* (4.9) *and* (4.10) *satisfy a.e. in* $[0, T]$ *the inequality*

$$\frac{d}{dt}\frac{1}{4}\,|z(t) - w(t)|^2 \leqslant -d(z(t), w(t)) \cdot (|\dot{\xi}(t)| + |\dot{\eta}(t)|). \tag{4.14}$$

Proof. Define

$$\tilde{z}(t)=\frac{z(t)+w(t)}{2}$$

and

$$n(\zeta)=\begin{cases} d(z(t), w(t))\,\dfrac{\zeta}{|\zeta|}, & \zeta\neq 0,\\ 0, & \zeta=0,\end{cases}$$

where d is defined in (4.8). We now test the variational inequality for z, namely

$$\langle \dot{\xi}(t), z(t)-\zeta\rangle \geqslant 0, \qquad \forall \zeta\in Z, \tag{4.15}$$

with

$$\zeta=\tilde{z}(t)+n(\dot{\xi}(t)),$$

and the variational inequality for w, namely

$$\langle \dot{\eta}(t), w(t)-\zeta\rangle \geqslant 0, \qquad \forall \zeta\in Z, \tag{4.15}$$

with

$$\zeta=\tilde{z}(t)+n(\dot{\eta}(t)).$$

We add the resulting inequalities and obtain

$$\left\langle \dot{\xi}(t)-\dot{\eta}(t), \frac{z(t)-w(t)}{2}\right\rangle - d(z(t), w(t))\cdot(|\dot{\xi}(t)|+|\dot{\eta}(t)|)\geqslant 0. \tag{4.17}$$

Since $\xi-\eta=w-z$, the lemma is proved. ∎

COROLLARY 4.1. *Let Z be γ-convex. Then the play nonlinearity $\mathscr{P}_Z$ is normal with the threshold $h=\operatorname{diam}(Z)$. In particular, property* (N2) *holds with q given by*

$$q_Z(\alpha)=e^{-4\gamma(\alpha-\operatorname{diam}(Z))}. \tag{4.18}$$

Proof. This follows from (4.7), because $\mathscr{P}_Z[z_0]\,u-\mathscr{P}_Z[z_1]\,u=\mathscr{S}_Z[z_1]\,u-\mathscr{S}_Z[z_0]\,u$. ∎

COROLLARY 4.2. *Let Z be a ball of the radius r. Then the stop and play nonlinearities are $2r$-normal and*

$$q(\alpha) = e^{1-\alpha/2r}, \qquad \alpha > 2r.$$

Proof. The ball with radius r is γ-convex with $\gamma = 1/8r$, as an elementary construction shows. ∎

In the application to elastoplasticity, the case when Z is a ball corresponds to the von Mises yield criterion.

When we use the play and the stop nonlinearities to construct more complex models of hysteresis, we may compute the corresponding q from the formulas given above and thus decide whether and for which threshold those models are normal. We do not carry out this investigation here.

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